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On calculation of exponential growth rates

by

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Abstract. Let $h : M \to M$ be a pseudo-Anosov homeomorphism on an orientable surface with boundary. For the induced homomorphism $h_* : \pi_1(M,*) \to \pi_1(M,*)$, we will simplify the formula on calculation of exponential growth rate. If we choose $\alpha \in \pi_1(M,*)$ which is not represented by a boundary parallel closed path, the simple limit $\lim_{n\to\infty}(1/n)\log|h_*^m(\alpha)|$ gives the exponential growth rate of h_*.

1. Introduction. In this article we will give a formula on calculation of exponential growth rates. Especially we are interested in exponential growth rates for homomorphisms on the fundamental group of a compact oriented surface M induced by homeomorphisms $h : M \to M$. Choosing a generating system of $\pi_1(M,*)$, we denote by $|\alpha|$ the word length of $\alpha \in \pi_1(M,*)$ with respect to this generating system. Recall that for a homomorphism ϕ of the group $\pi_1(M,*)$, the exponential growth rate $\mathrm{EGR}(\pi_1(M,*),\phi)$ is given by the formula

$$\mathrm{EGR}(\pi_1(M,*),\phi) = \sup_{\alpha\in\pi_1(M,*)} \limsup_{m\to\infty} \frac{1}{m}\log|\phi^m(\alpha)|.$$

Note that the definition of the exponential growth rate is independent of the choice of generating systems of the group $\pi_1(M,*)$.

In the case when a homeomorphism h is pseudo-Anosov and M has non-empty boundary, this formula is slightly simplified for suitable choice of an element $\alpha \in \pi_1(M,*)$.

Theorem. *Assume that h is a pseudo-Anosov homeomorphism on a compact oriented surface M with non-empty boundary. If an element $\alpha \in \pi_1(M,*)$ is not represented by a boundary parallel closed path, then*

$$\mathrm{EGR}(\pi_1(M,*),h_*) = \lim_{m\to\infty} \frac{1}{m}\log|h_*^m(\alpha)|.$$

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Note that even if h does not preserve the base point, the exponential growth rate is well defined, because we have an isomorphism $\pi_1(M, h^m(*)) \to \pi_1(M, *)$ which is uniquely determined up to inner automorphism of $\pi_1(M, *)$.

In §.2 for a pseudo-Anosov homeomorphism h we will construct a graph so called a train track [3] from the unstable and stable foliations [1] associated to h. In §.3 using the train track we will give a lemma to describe the topology of the image of a closed path under an iteration of h, and then prove the theorem.

2. Train track. As the proof of 4.1.Theorem [2], remove from M a neighborhood of the singularity of the unstable foliation associated to h and denote by M_1 the obtained holed surface. We make a quotient space of M_1 by collapsing each connected component of the intersection of M_1 with the stable leaves to a single point. Then we obtain a train track G_1, and furthermore $h : M \to M$ and the projection $\pi : M_1 \to G_1$ determine a homotopy equivalence $f_1 : G_1 \to G_1$. Note that G_1 may be viewed as to be embedded in M. Let k be the number of singular points of the unstable foliation which are in the interior of M, and let F_k be the free subgroup of $\pi_1(G_1, *)$ which is defined by the boundary circles of M_1 corresponding to the inner singularity. Then if we regard $\pi_1(M, *)$ as a subgroup of $\pi_1(G_1, *)$, it is obvious that $\pi_1(G_1, *) = F_k * \pi_1(M, *)$. Let us choose a minimal generating system of $\pi_1(G_1, *), \alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_{n+k}$, such that $\pi_1(M, *) = \langle \alpha_1, \cdots, \alpha_n \rangle$ and $F_k = \langle \alpha_{n+1}, \cdots, \alpha_{n+k} \rangle$, and let $p : \pi_1(G_1, *) \to \pi_1(M, *)$ denote the projection.

Recall that the neighborhood of the singularity chosen in the previous paragraph is the disjoint union of polygons (where we mean by a polygon the closed region bounded by it) with edges transverse to the stable leaves and vertices on the unstable separatrices. Furthermore for each single inner singular point we choose a distinct single polygon, and for all singular points on each single boundary component of M we choose a single polygon with boundary in Int$M$ and with hole which is bounded by the boundary component itself. If we choose polygons corresponding to the inner singularity as to be sufficiently small relative to the neighborbood of ∂M with respect to the sum of the length of boundary edges measured by the transvers measure, then collapsing each subgraph of G_1 corresponding to the inner singularity, we obtain a graph G homotopy equivalent to M. Let us denote this projection by $q : G_1 \to G$. Then clearly $q_* = p$, and $f_1 : G_1 \to G_1$ projects down to a homotopy equivalence $f : G \to G$. Furthermore we may assume that any vertex of G_1 has three prongs.

3. Proof of Theorem. Using the train track constructed in the previous section we will show the following lemma.

Lemma. *Under the same condition as Theorem, there exists a positive integer m such that for the generating system chosen in the previous section the reduced word of $h_*^m(\alpha)$ includes α_j or its inverse $\bar{\alpha}_j$ for all j.*

Proof. Let a be the closed path in G_1 which determines α. We will construct a closed path $\tilde{a}$ in M which is homotopic to a and which is the concatenation of segments lying in the stable, unstable separatrices and boundary circles, and with one endpoint the singularity. More precisely in the following paragraph we will choose $\tilde{a}$ as the concatenation $s_0 \cdot u_1 \cdot s_1 \cdot u_2 \cdot s_2 \cdots u_l \cdot s_l$, where each s_i is in a stable separatrix, or in the union of a boundary circle and a stable separatrix emanating from it, perhaps s_0 is trivial, *i.e.* a vertex, and each u_i is in an unstable separatrix, and the initial points of u_j and the terminal points of s_j are singular points.

By the assumption on G_1, we can detect which singular point a vertex v of G_1 corresponds to, and more precisely which vertex of removed polygon it corresponds to. Assume that a passes an edge ε with end points vertices v_1 and v_2 in the direction from v_1 to v_2, and assume that v_j correspond to distinct singular points x_j for $j = 1, 2$. By construction $\pi^{-1}(\operatorname{Int}\varepsilon)$ is a foliated rectangle, and we can uniquely determine an unstable separatrix ζ at x_1 which goes into $\pi^{-1}(\operatorname{Int}\varepsilon)$ and a stable separatrix ξ at x_2 which intersects with ζ at y in $\pi^{-1}(\pi(P_2))$ where P_2 denotes the boundary of the removed polygon for x_2. Then for paths u_j and s_j, with suitable index j, we choose the paths in ζ and ξ bounded by x_1 and y, and y and x_2 respectively. If $x_1 = x_2$, we do not need to choose u_j and s_j for the edge ε, and if x_1 and x_2 lie in the same boundary component, we also do not need to choose u_j and s_j, but in this case we replace the last chosen s_{j-1} by the concatenation of it with the arc η in the boundary bounded by x_1 and x_2 if $j \geq 2$, and if $j = 1$, we choose η as s_0. Note that by the assumption on a, there exists a non-trivial, *i.e.* not reduced to a vertex, u_j.

For each vertex v of G_1 let us choose an edge path δ_v in G_1 connecting it to the base point $*$. Then for any path γ in G_1 with endpoints v_0 and v_1 vertices, concatenating the reverse path $\bar{\delta}_{v_0}$ to the one chosen for v_0, γ, and the path δ_{v_1} chosen for v_1 in this order, we obtain a closed path. Let us denote by $\langle\gamma\rangle$ the element of $\pi_1(G_1, *)$ determined by this closed path.

Assume that $u_i \cdot s_i$ has the initial point x_0^i and the terminal point x_1^i. For $\epsilon = 0$ and 1 let B_ϵ^i be the subgraphs of G_1 corresponding to the boundary components of M_1 with respect to x_ϵ^i. Pushing out u_i from the polygons chosen for the singularity as sliding along stable leaves and keeping the terminal point in the sector bounded by two adjacent unstable separatrices, we obtain a path u_i' and then this projects down to a path $\tilde{u}_i$ in G_1. For each $i = 1, 2, \cdots, l$ the terminal point $\tilde{x}_1^i$ of $\tilde{u}_i$ and the initial point $\tilde{x}_0^{i+1}$ of $\tilde{u}_{i+1}$ are in $B_1^i = B_0^{i+1}$, and let us choose a path $\tilde{t}_i$ in B_1^i which connects $\tilde{x}_1^i$ with $\tilde{x}_0^{i+1}$, where $l + 1$ is viewed as 1. Then the concatenation $\tilde{b} = \tilde{u}_1 \cdot \tilde{t}_1 \cdot \tilde{u}_2 \cdot \tilde{t}_2 \cdots \tilde{u}_l \cdot \tilde{t}_l$ is a closed path in G_1, and it defines the element α again even though the choice of $\tilde{t}_i$ is not unique. Replacing $\tilde{a}$, if necessary, we may assume that $\tilde{b}$ has no back track, and more precisely we may assume that if in a neighborhood of the singular point x_0^{i+1} u_i is pushed to a separatrix χ_i along stable leaves, then the separatrix χ_{i+1} in which u_{i+1} lies is distinct to χ_i.

Set $u_i(m) = h^m(u_i)$ and $s_i(m) = h^m(s_i)$. For paths $u_i(m)$ we perform the same construction as to obtain $\tilde{u}_i$ from u_i, and then we obtain paths $\tilde{u}_i(m)$. Since h maps the singularity into itself, $\tilde{u}_i(m)$ is well determined. We choose paths $\tilde{t}_i(m)$ in the same way as to choose $\tilde{t}_i$. Set $\tilde{b}(m) = \tilde{u}_1(m) \cdot \tilde{t}_1(m) \cdot \tilde{u}_2(m) \cdot \tilde{t}_2(m) \cdots \tilde{u}_l(m) \cdot \tilde{t}_l(m)$. By construction $q(\tilde{b}(m)) = f^m(q(\tilde{b}))$ and $q(\tilde{u}_i(m)) = f^m(q(\tilde{u}_i))$. Replace $\tilde{u}_i(m)$ by the inner most edge path $\widehat{u}_i(m)$. Then by construction we have $q(\tilde{b}(m)) = q(\widehat{u}_1(m)) \cdot q(\widehat{u}_2(m)) \cdots q(\widehat{u}_l(m))$, and $h_*^m(\alpha) = q_*(\langle \widehat{u}_1(m) \rangle \cdot \langle \widehat{u}_2(m) \rangle \cdots \langle \widehat{u}_l(m) \rangle)$ up to conjugacy.

Now we will assert that a reduced word to represent $h_*^m(\alpha)$ is produced by formally making a product $\langle \widehat{u}_1(m) \rangle \cdot \langle \widehat{u}_2(m) \rangle \cdots \langle \widehat{u}_l(m) \rangle$ and removing letters $\alpha_{n+1}, \alpha_{n+2}, \cdots, \alpha_{n+k}$ from it. Then this completes the proof, because each unstable separatrix is dense in M, and thus for a large m $\widehat{u}_i(m)$ laps each edges sufficiently large times. Therefore the above assertion implies that the reduced word to represent $h_*^m(\alpha)$ includes any α_j or its inverse for $j = 1, 2, \cdots, n$.

We denote by $\langle\langle \widehat{u}_i(m) \rangle\rangle$ the word obtained from $\langle \tilde{u}_i(m) \rangle$ by removing letters $\alpha_{n+1}, \alpha_{n+2}, \cdots, \alpha_{n+k}$. Let γ_i and γ_{i+1} be the last letter and the first letter of $\langle\langle \widehat{u}_i(m) \rangle\rangle$ and $\langle\langle \widehat{u}_{i+1}(m) \rangle\rangle$. Since h maps a singular point with k unstable separatrices to a singular point of the same type, by construction we have that $\gamma_i \neq \overline{\gamma}_{i+1}$, and thus cancellation may be done only in each word $\langle\langle \widehat{u}_i(m) \rangle\rangle$. Suppose by contradiction that the word $\langle\langle \widehat{u}_i(m) \rangle\rangle$ includes letters to be cancelled. Then it follows that $\tilde{u}_i(m)$ has a back track, but this is impossible because $\tilde{u}_i(m)$ is an immersed curve in G_1. This completes the proof. $\square$

We have done the all preparations to prove the theorem. To complete the proof, only a little bit argument is needed.

Proof of Theorem. As shown in the proof of Lemma, for a sufficiently large $\bar{m}$ $\langle \widehat{u}_i(\bar{m}) \rangle$ has a word α_j or $\bar{\alpha}_j$ for any $j, 1 \leq j \leq n$, and when we remove $\alpha_j, n+1 \leq j$, any other cancellation does not occur. Therefore we have

$$\limsup_{m \to \infty} \frac{1}{m} \log |h_*^m(\alpha)| \geq \limsup_{m \to \infty} \frac{1}{m} \log |h_*^m(\alpha_j)|$$

Furthermore by the above argument $|h_*^m(\alpha_j)|$ increase monotonically, and thus in the above inequality we may replace the limit supremums by simple limits :

$$\lim_{m \to \infty} \frac{1}{m} \log |h_*^m(\alpha)| \geq \lim_{m \to \infty} \frac{1}{m} \log |h_*^m(\alpha_j)|.$$

Since the contrary inequality is obvious, we complete the proof of Theorem. $\square$

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